

MTH250 - CALCULUS

Lecture No. 29

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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LECTURE 29

Line integrals II

29.1 Integrating vector fields along curves

Let a curve be parameterized by $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, then,

$$d\mathbf{r} = dx\hat{i} + dy\hat{j}.$$

Therefore, for a vector field $\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$

$$\int_C f(x, y)dx + g(x, y)dy = \int_C (f(x, y)\hat{i} + g(x, y)\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Definition 29.1. If $\mathbf{F}$ is a continuous vector field and C is a smooth oriented curve then the line integral of $\mathbf{F}$ along C is $\int_C \mathbf{F} \cdot d\mathbf{r}$. If the curve is parameterized by $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, and $\mathbf{F}(x, y) = f(x(t), y(t))\hat{i} + g(x(t), y(t))\hat{j}$. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Example 29.1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \cos x\hat{i} + \sin x\hat{j}$ and where C is the given oriented curve

$$C: \mathbf{r}(t) = -\frac{\pi}{2}\hat{i} + t\hat{j} \quad (1 \leq t \leq 2) \quad \text{and} \quad C: \mathbf{r}(t) = t\hat{i} + t^2\hat{j} \quad (-1 \leq t \leq 2)$$

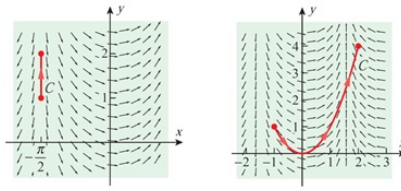


Figure 29.1: Left: $C: \mathbf{r}(t) = -\frac{\pi}{2}\hat{i} + t\hat{j} \quad (1 \leq t \leq 2)$. Right: $C: \mathbf{r}(t) = t\hat{i} + t^2\hat{j} \quad (-1 \leq t \leq 2)$

Solution.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_1^2 (-\hat{j}) \cdot \hat{j} dt = \int_1^2 (-1) dt = -1.$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_{-1}^2 (\cos t \hat{i} + \sin t \hat{j}) \cdot (\hat{i} + 2t\hat{j}) dt \\ &= \int_{-1}^2 (\cos t + 2t \sin t) dt = (-2t \cos t + 3 \sin t) \Big|_{-1}^2 \\ &= -2 \cos 1 - 4 \cos 2 + 3(\sin 1 + \sin 2) \approx 5.83629. \end{aligned}$$

□

If t is an arc length parameter *i.e.* $t = s$ then $\mathbf{T} = \mathbf{r}'(s)$ is the unit tangent vector field along C . Then;

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{r}'(s) ds = \int_a^b \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

In other words, the integral of a vector field along a curve has the same value as the integral of the tangential component of the vector field along the curve.

If θ is the angle between $\mathbf{F}$ and $\mathbf{T}$ then $\mathbf{F} \cdot \mathbf{T} = |\mathbf{F}||\mathbf{T}| \cos \theta$. If $\mathbf{F} \neq \mathbf{0}$ then the sign of $\mathbf{F} \cdot \mathbf{T}$ will depend on the angle between the direction of $\mathbf{F}$ and the direction of C . It will be 0 if $\mathbf{F}$ is normal to C , negative if it has opposite direction then C and positive if both have the same direction.

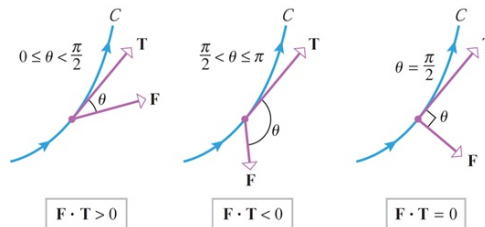


Figure 29.2: Orientation of $\mathbf{F}$ with respect to C .

Example 29.2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = -y\hat{i} + x\hat{j}$ and where C is the oriented curve (a) $C: x^2 + y^2 = 3$ ($0 \leq x, y$) and (b) $C: \mathbf{r}(t) = t\hat{i} + 2t\hat{j}$ ($0 \leq t \leq 1$)

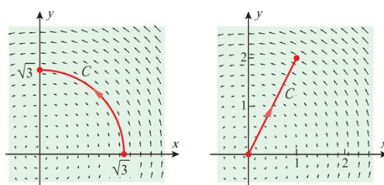


Figure 29.3: $\mathbf{F}(x, y) = -y\hat{i} + x\hat{j}$. Left: $C: x^2 + y^2 = 3$ ($0 \leq x, y$). Right: $C: \mathbf{r}(t) = t\hat{i} + 2t\hat{j}$ ($0 \leq t \leq 1$)

29.1.1 Work done by a force

Definition 29.2 (Work done). Suppose that under the influence of a continuous force field $\mathbf{F}$ a particle moves along a smooth curve C and that C is oriented in the direction of motion of the particle. Then the work performed by the force field on the particle is

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Example 29.3. Find the work done by the force field $\mathbf{F}(x, y) = x^2\hat{i} - xy\hat{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\hat{i} + \sin t\hat{j}$, $0 \leq t \leq \frac{\pi}{2}$.

Solution. Since, $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\hat{i} - \cos t \sin t\hat{j}$$

and

$$\mathbf{r}'(t) = -\sin t\hat{i} + \cos t\hat{j}.$$

Therefore, the work done is

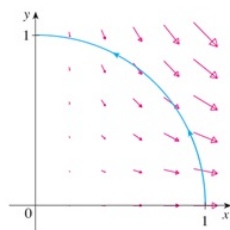


Figure 29.4: $\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\hat{i} - \cos t \sin t\hat{j}$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{\pi/2} (-2 \cos^2 t \sin t) dt = 2 \frac{\cos^2 t}{3} \Big|_0^{\pi/2} = -\frac{2}{3}$$

□

29.2 Fundamental theorem of Calculus

Definition 29.3. The parametric curve C in a line integral is called the *path of integration*.

Definition 29.4 (Conservative field). A force field is $\mathbf{F}$ called *conservative* if there exists a scalar function ϕ , called the *potential*, such that the

$$\mathbf{F}(x, y) = \nabla \phi(x, y).$$

Theorem 29.5 (The fundamental theorem of line integrals).

Suppose that

$$\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$$

is a conservative vector field in some open region D containing the points (x_0, y_0) and (x_1, y_1) and that $f(x, y)$ and $g(x, y)$ are continuous in this region. If

$$\mathbf{F}(x, y) = \nabla \phi(x, y)$$

and if C is any piecewise smooth parametric curve that starts at (x_0, y_0) , ends at (x_1, y_1) , and lies in the region D , then

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

or, equivalently,

$$\int_C \nabla \phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

Proof. Suppose that C is given parametrically by $x = x(t)$, $y = y(t)$, ($a \leq t \leq b$), so that the initial and final points of the curve are

$$(x_0, y_0) = (x(a), y(a)) \quad \text{and} \quad (x_1, y_1) = (x(b), y(b)).$$

Since $\mathbf{F}(x, y) = \nabla \phi$, it follows that

$$\mathbf{F}(x, y) = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j},$$

so

$$\begin{aligned} \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} &= \int_C \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \int_a^b \left[\frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} \right] dt \\ &= \int_a^b \frac{d}{dt} [\phi(x(t), y(t))] dt = \phi(x(t), y(t)) \Big|_{t=a}^t=b \\ &= \phi(x(b), y(b)) - \phi(x(a), y(a)) = \phi(x_1, y_1) - \phi(x_0, y_0). \end{aligned}$$

□

29.2.1 Independence of path

The fundamental theorem of Calculus for line integrals points out that the value of a line integral of a conservative vector field along a piecewise smooth path is independent of the path, that is, the value of the integral depends on the endpoints and not on the actual path C .

Accordingly for a line integral of conservative vector field $\mathbf{F}$ is given by

$$\int_{(x_0, y_0)}^{(x_1, y_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0)}^{(x_1, y_1)} \nabla \phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0).$$

Example 29.4. Confirm that the force field

$$\mathbf{F}(x, y) = y\hat{i} + x\hat{j}$$

is conservative by showing that $\mathbf{F}$ is the gradient of $\phi(x, y) = xy$. Use fundamental theorem of line integrals to evaluate $\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r}$.

Solution. We have

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} = y\hat{i} + x\hat{j}.$$

Therefore,

$$\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = \phi(1, 1) - \phi(0, 0) = 1 - 0 = 1.$$

□

29.3 Integration along closed path

Definition 29.6 (Connected sets).

We mean by a connected set D , that any two points in D can be joined by some piecewise smooth curve that lies entirely in D . Informally, D is connected if it does not consist of two or more separate pieces.

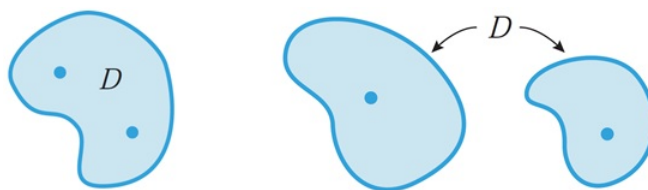


Figure 29.5: Connected (left) and not connected (right).

Theorem 29.7. If $f(x, y)$ and $g(x, y)$ are continuous on some open connected region D then the following statements are equivalent (all true or all false):

1. $\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$ is conservative vector field on the region D .
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth closed curve C in D .
3. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from any point P in D to any point Q in D for every piecewise smooth curve C in D .

29.4 Test for conservative field

Definition 29.8 (Simply connected sets).

We say that a parametric curve is simple if it does not intersect itself between its endpoints. A simple parametric curve may or may not be closed.

We say that a connected set S in $2D$ is simply connected if no simply closed curve in S encloses points that are not in S . Informally, a connected set S is simply connected if it has no holes.

A connected set with one or more holes is said to be multiply connected.

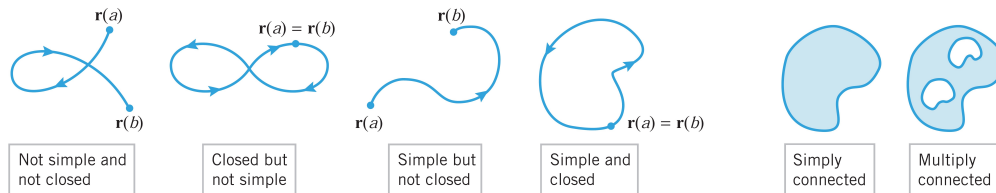


Figure 29.6:

Theorem 29.9 (Conservative field test). If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open region D , and if the vector field

$$\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$$

is conservative on D , then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad (29.1)$$

at each point in D . Conversely, if D is simply connected and Equation 29.1 holds at each point in D , then

$$\mathbf{F}(x, y) = f(x, y)\hat{i} + g(x, y)\hat{j}$$

is conservative.

Example 29.5. Determine whether $\mathbf{F}(x, y) = (y + x)\hat{i} + (y - x)\hat{j}$ is conservative or not.

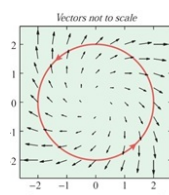


Figure 29.7: Vector field $\mathbf{F}(x, y) = (y + x)\hat{i} + (y - x)\hat{j}$.

Solution. Let $f(x, y) = y + x$ and $g(x, y) = y - x$. Then,

$$\frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = -1$$

Thus, there are no points in the xy -plane at $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. Hence, $\mathbf{F}$ is not a conservative field on any open set. $\square$

Example 29.6. Let $\mathbf{F}(x, y) = e^y\hat{i} + xe^y\hat{j}$ denote a force field in the xy -plane.

1. Verify that the force field $\mathbf{F}$ is conservative on the entire xy -plane.
2. Find the work done by the field on a particle that moves from $(1, 0)$ to $(-1, 0)$ along the semicircular path C .

Solution.

1. For a given field, we have $f(x, y) = e^y$ and $g(x, y) = xe^y$. Thus,

$$\frac{\partial}{\partial y}(e^y) = e^y = \frac{\partial}{\partial x}(xe^y)$$

and hence $\mathbf{F}$ is conservative on the entire xy -plane.

2. $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C e^y dx + xe^y dy$. However, taking advantage of the fact that the field is conservative and the integral is independent of path. Thus, we write

$$W = \int_{(1,0)}^{(-1,0)} e^y dx + xe^y dy = \phi(-1, 0) - \phi(1, 0)$$

we can find ϕ by integrating either of the equations

$$\frac{\partial \phi}{\partial x} = e^y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = xe^y.$$

We will integrate the first one. We obtain

$$\phi = \int e^y dx = xe^y + k(y).$$

Differentiating this equation with respect to y and using the second equation yields

$$\frac{\partial \phi}{\partial y} = xe^y + k'(y) = xe^y$$

from which it follows that $k'(y) = 0$ or $k(y) = K$. Thus,

$$\phi = xe^y + K$$

and hence

$$W = \phi(-1, 0) - \phi(1, 0) = (-1)e^0 - 1e^0 = -2.$$

□